ANTIQUITATES MATHEMATICAE
Vol. 17(1) 2023, p. 5–27
doi: 10.14708/am.v17i1.7160

J. SHILLETO (Richmond, CA, USA)
JAMES T. SMITH (San Francisco, CA, USA)

Stairs: The *Only* Way to Go

Abstract. In plane Euclidean geometry, any polygonal regions Q, Q' with equal area can be dissected into the same number n of pairwise congruent subregions. This is easy to see if Q is a unit square, Q' is an $(m+1)/m$ by $m/(m+1)$ rectangle for some integer m, and $n = m+1$. During the 1920s the renowned logician Alfred Tarski, then teaching elementary geometry in Warsaw, noticed a different dissection for the same Q, Q' with $n = 3$. He reported it in a pioneering journal for secondary-school teachers and students, then asked, are there such dissections with $n = 2$? Henryk Moese, a schoolteacher, responded yes, in some cases: m-step "staircase" dissections are possible for these dimensions. Moese conjectured that the *only* such dissections of Q, Q' are those. Tarski reported that this had been confirmed, but the proof was too complex to publish. Decades later, Tarski presented this material to general academic audiences that included the present authors. We regarded Moese's conjecture as a challenge, devised a way to verify it, and present that argument here. Using only elementary geometry, with many figures, it suggests what methods Tarski might have expected for solving other problems in that journal, but is unlike any arguments in related literature. Information is included about Polish efforts in the 1930s to improve secondary-school instruction, and about the role of Tarski's research seminar.

2010 Mathematics Subject Classification: 01A60 (01A70, 01A73).

Key words and phrases: Mathematics and mathematicians of the 20th century in Poland, Józef Piłsudski University of Warsaw, Alfred Tarski, *Parametr*, elementary geometry.

Preface. This paper is about an application of elementary plane geometry to a simple but rather unusual problem posed in 1931 by the renowned logician Alfred Tarski, who had been studying the axiomatization of geometry intensively in connection with his research on logic and with his duties as a secondary-school geometry teacher in Warsaw. The problem would test the formulation of such axioms as well as stimulate the imagination and challenge the proficiency of geometry students and teachers. A year later, Tarski claimed that the problem had been solved by colleagues, but a full solution was too involved to be published. Decades later, Tarski repeated that in lectures. The present authors, who attended, offer a complete solution here. Our techniques are probably like what Tarski had in mind, but are quite different from methods usually presented in schools. Main features of our solution are its requirement for ultraprecise statement of the problem, and the organization of its proof. These lie outside the mainstream of elementary geometry instruction. We are impressed that in 1930 Tarski and his colleagues were evidently concerned with methods that foreshadowed those developed decades later for computational geometry.

1. Introduction. According to a fundamental principle of plane Euclidean geometry, any polygonal regions Q, Q' with equal area can be dissected into the same number n of pairwise congruent subregions:

$$\begin{aligned} &Q = Q_1 \cup \cdots \cup Q_n \text{ and } Q' = Q'_1 \cup \cdots \cup Q'_n \\ &\text{with } Q_i \cong Q'_i \text{ for } i = 1, \ldots, n. \end{aligned} \tag{1}$$

For example, if m is a positive integer, then a unit square Q and a rectangle Q' with dimensions $(m+1)/m$ and $m/(m+1)$ have the same area and figure 1 shows such a dissection with $n = m + 1$.

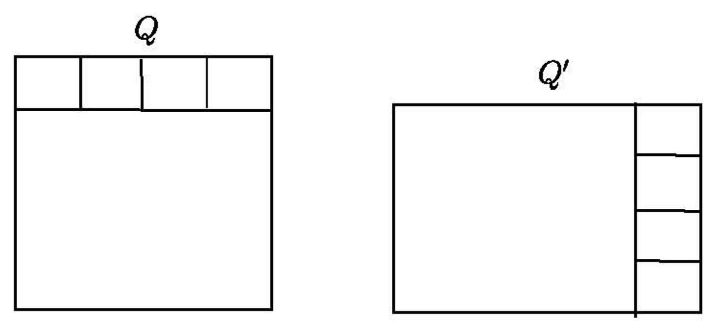


Fig. 1. Dissections into $n = 5$ rectangles, congruent in pairs $(m = 4)$

Are there other ways to show that two polygons have equal area? The second author (Smith) recalls considering in tenth grade, 1955, the question "Can you think of a way to transform a pentagon into a quadrilateral?" from the 1937 textbook by John C. Stone and Virgil S. Mallory [13, page 362].

Alfred Tarski earned the PhD in Warsaw in 1924 and immediately published a major contribution [14] to area theory. He was starting his career several years after the newly independent Poland had greatly expanded its university faculties. Hardly any university positions remained open. Teaching geometry at a Warsaw secondary school, he analyzed a proof of the Pythagorean theorem in the 1926 textbook by Władysław Wojtowicz [21, page 168] and noticed different dissections for the same Q, Q' with $n = 3$, shown here in figure 2. Readers can verify that these are possible whenever $0 < y < 1$.

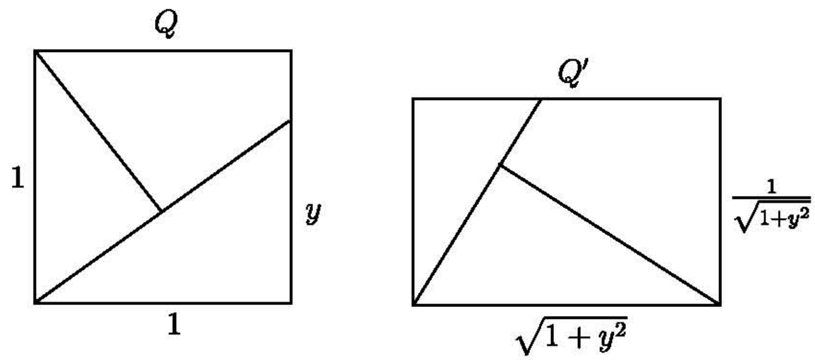


Fig. 2. Tarski's dissections into $n = 3$ polygons.

During the 1920s, Tarski and other Polish mathematicians lobbied for greater support for secondary-school mathematics. Responding in 1930, Antoni M. Rusiecki, a curriculum developer and teacher trainer, founded the monthly mathematical journal *Parametr* for secondary-school teachers and their best students. In 1931 he started its supplement, *Młody matematyk* (*Young Mathematician*) specifically for students.[1] Tarski contributed an elegant article that included the question whether there is an even more economical dissection of Q, Q' with $n = 2$. More generally, he defined the *degree of equivalence* of any two plane polygonal regions to be the *smallest* number n of subregions in a dissection of the form (1), then asked, what is the degree of equivalence of a square Q and a nonsquare rectangle Q' with the same area? Figure 2 shows that the degree is at most three. Can they be dissected into *two* polygonal subregions to show that the degree is actually two?[2]

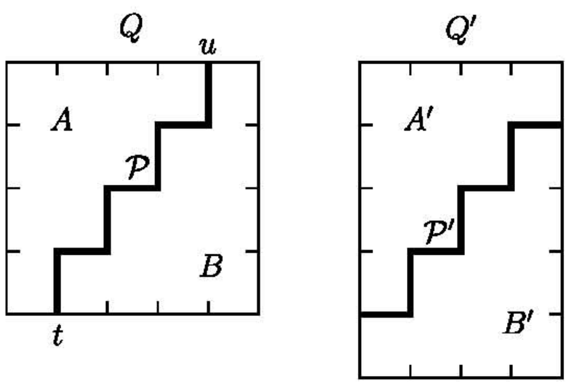
[1]For more information on those efforts, see the notes at the end of this paper.

[2]See Tarski's article [17]: figure 2 and the question are on pages 37 and 40–41.

Fig. 3. Alfred Tarski around 1930

Fig. 4. Henryk Moese in 1935

Henryk Moese, a schoolteacher in the small Polish city Kępno, responded a year later: yes, in some cases. The *m-step staircase* dissections in figure 5 are possible in the cases depicted there, with $m = 1, 2, \ldots$. Moese conjectured that the *only* dissections of Q, Q' each into two pairwise congruent polygonal subregions are these. That would imply that the degree of equivalence of Q, Q' is three unless the dimensions of Q' are $m/(m+1)$ by $(m+1)/m$ for some integer m, in which case the degree is two.



Q' must have dimensions $\frac{m}{m+1}$ by $\frac{m+1}{m}$.
Here, $m = 4$.

$\mathcal{P} = \partial_{\neg Q} A = \partial_{\neg Q} B$
$\mathcal{P}' = \partial_{\neg Q'} A' = \partial_{\neg Q'} B'$

Fig. 5. Moese's m-step staircase dissections into $n = 2$ polygons [3]

Neither Tarski nor Moese mentioned any specific background source for this investigation except Wojtowicz's textbook [21]. That was an attempt to streamline for Polish students the famous text [9] by Federigo Enriques and Ugo Amaldi, which Tarski had cited [14, page 47] to support his research in area theory. The Italians' text was mathemati-

[3]See Moese's article [12, page 308].

cally correct and delightful reading for professional mathematicians, but made no allowance for those whose reading and study skills had not yet attained that level. Wojtowicz's book was a slight improvement.[4]

Alongside his secondary-school teaching, Tarski remained deeply involved with research. In 1932 he reported that his University colleagues Adolf Lindenbaum and Zenon Waraszkiewicz had verified Moese's conjecture.[5] Tarski wrote [18, page 312] that their proof "is somewhat complicated and requires some subtle methods of reasoning." Tarski did not include that in his article. Indeed, it is not immediately clear how one should argue that the *only* way to achieve the desired dissection is to use a staircase; moreover, describing the process accurately would require many tedious details.

Fig. 6. Adolf Lindenbaum around 1922

Fig. 7. Zenon Waraszkiewicz around 1930

Tarski published no more on this subject but did present it in elegant lectures to mathematically literate general audiences. The present authors attended those events. Tarski's assessment of Moese's conjecture posed a challenge. We devised a way to verify it, and present that argument here. It suggests what methods Tarski might have expected to be used for solving some other problems that he posed in the same journal, but is unlike any argument that we have encountered in related literature. It requires precise formulation somewhat like what is needed

[4]The 1937 Stone & Mallory text [13] mentioned earlier was far more accessible, but ignored many important considerations. Unlike most other texts, however, it encouraged students to discover geometric properties on their own. Tarski himself contributed to a textbook: see the notes at the end of this paper.

[5]For information about these mathematicians see section 5 of this paper.

for recent developments in computational geometry.[6]

2. Precision. To verify Moese's conjecture — that the *only* dissections of a square Q and a nonsquare rectangle Q' each into two pairwise congruent polygonal subregions are the staircases of figure 5 — requires phrasing several ideas more precisely than usual: congruence, polygonal region, dissection, and the meaning of *only*. This section will provide that precision. Readers may want to sketch those notions and the associated ones defined and described next.

Two point sets are *congruent* ($\cong$) just when they are related by a *motion*, or *isometry*. Isometries are the distance-preserving transformations of the set of all points: translations, rotations, reflections, and glide reflections. Isometries that agree at three noncollinear points must agree everywhere.[7]

The *segment* $|pq|$ between *end* points p, q contains both; its other points are called *internal*. A (*simple*) *polygonal path* is a finite linearly or cyclically ordered set of *edge* segments, such that two edges intersect just when they are consecutive, noncollinear, and share a common end, which is called a *vertex*. The distinction between the orderings is revealed by the context and by the usage of words such as *consecutive*. Cyclically ordered paths are called (*closed*) *polygons*. For example, the square and rectangle in figure 5 represent closed paths, each consisting of four segments in cyclic order. The staircases depicted by heavy lines represent nonclosed paths, each consisting of seven segments in linear order. The *interior* and *exterior* of a polygon are the sets of points q, not in any edge, such that some ray starting at q and not containing any vertex intersects the edges at an odd or at an even number of points, respectively. The polygon and its interior and exterior are disjoint, and their union is the entire plane. The union A of a polygon and its interior is called a *polygonal region*; the polygon itself is called the *boundary* ∂A of A. Figure 8 shows a shaded polygonal region A, its boundary ∂A, and interior and exterior points q and q'. All these

[6] Tarski lectured to high-school students around 1967 in Berkeley and in 1970 to a general audience in Regina, Canada. For his problems and some solutions, see chapter 12 of the book [11], which is devoted to Tarski's early life and work. For computational geometry, compare the 2011 text [5] by Satyan Devadoss and Joseph O'Rourke.

[7] See Amaldi [1], to which Tarski referred in his research paper [14, page 59], or the 1941 book by Richard Courant and Herbert Robbins [3, chapter V]. Few resources treat this subject with the precision appropriate for the present study; reference [5] or [10] may be helpful.

notions are preserved by every isometry α: for example, the image set $\alpha[A]$ is a polygonal region and $\alpha[\partial A] = \partial\alpha[A]$.

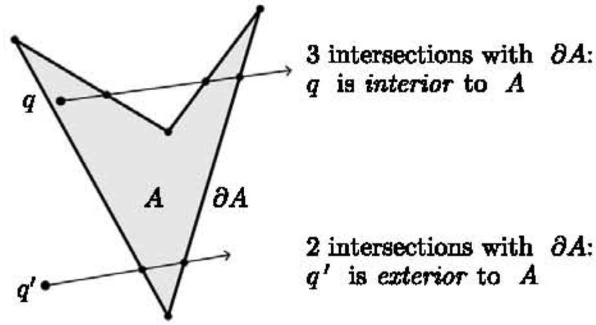


Fig. 8. Polygonal region A with boundary ∂A and interior and exterior points

In this paper, boundaries have more structure than what is needed for most studies: they are families of segments, not sets of points. The following concept is employed in stating Moese's result and his conjecture: for a subregion A of a polygonal region Q as in figure 5, $\partial_{\neg Q}A$ will denote the family of all segments of ∂A whose points do not all fall in segments in the boundary of Q.[8]

- **Moese's Result.** Figure 5 depicts *m-step staircase* dissections $Q = A \cup B$ and $Q' = A' \cup B'$ of square and nonsquare rectangles into polygonal subregions with disjoint interiors, such that $A \cong A'$, $B \cong B'$, and the sets $\mathcal{P} = \partial_{\neg Q}A = \partial_{\neg Q}B$ and $\mathcal{P}' = \partial_{\neg Q'}A' = \partial_{\neg Q'}B'$ (dark lines) are polygonal paths.

- **Conjecture**. Moreover, for any such pair of dissections the sets $\mathcal{P} \cup \partial Q$ and $\mathcal{P}' \cup \partial Q'$ are congruent to the corresponding sets for some m-step staircase dissections.

The discussion in the following sections refers to isometries α and β that map A onto A' and B onto B', respectively.

3. Orientation. The main task of this paper is to formulate in the ordinary language of elementary geometry an argument that can be made into a proof of a formal statement of Moese's conjecture in an axiomatic theory. The previous section 2 has clarified our terminology.

[8]This can be phrased $\partial_{\neg Q}A = \{S \in \partial A : S \not\subseteq \bigcup\partial Q\} = \partial A - \mathfrak{P}\bigcup\partial Q$, using $\bigcup$ and $\mathfrak{P}$ for union and power set.

The organization of the argument goes somewhat beyond the usual level of elementary geometry courses. This section 3 begins the analysis of arbitrary dissections $Q = A \cup B$ and $R_x = A' \cup B'$ into polygonal subregions whose interiors are disjoint, such that $A \cong A'$ and $B \cong B'$, and such that the sets $\mathcal{P} = \partial_{\neg Q} A = \partial_{\neg Q} B$ and $\mathcal{P}' = \partial_{\neg Q'} A' = \partial_{\neg Q'} B'$ are polygonal paths that separate the pairs of subregions. Some partial results about the orientation of the paths within Q and R_x are derived. The following section 4 on construction shows that the paths must be staircases like Moese's. To check that a particular axiom system for elementary geometry actually supports this argument would require more elaborate formalization.

Our argument starts with two lemmas that result from analytic-geometry calculations. The first is simple. Let $0 < x < 1$ and consider a rectangle R_x with *horizontal* edges of length $x < 1$ and *vertical* edges of length $1/x > 1$.

LEMMA 3.1 *The diagonals of R_x have length*

$$\sqrt{x^2 + \frac{1}{x^2}} = \sqrt{\left(x - \frac{1}{x}\right)^2 + 2} > \sqrt{2}.$$

Thus, no isometry maps opposite corners of R_x into a unit square Q.

Next, we consider how certain L-shaped portions of the boundary of Q might be mapped into R_x. The union of perpendicular segments of lengths 1 and z that share a common end will be called an L_z. The second segment will be dashed in figures, and the other two ends will be called *free*.

LEMMA 3.2 *The only case in which an L_1 lies inside R_x is depicted in figure 9a.*

PROOF (extending over this paragraph, figures 9b to 10b, and the following three paragraphs). Consider an arbitrary L_z in R_x. Its unit segment is too long to be horizontal, so the horizontal lines H and J through its free end and through its vertex must be distinct. If the other free end q should fall between H and J as in figure 9b, then the L_z can be translated and rotated within R_x so that its unit and dashed segments become vertical and horizontal. In that case, $z \leq x < 1$. In particular, no L_1 can lie in R_x in this position.

In the other case, q will fall on the sides of H and J opposite one of the horizontal edges of R_x; call that the *bottom* edge. Moreover,

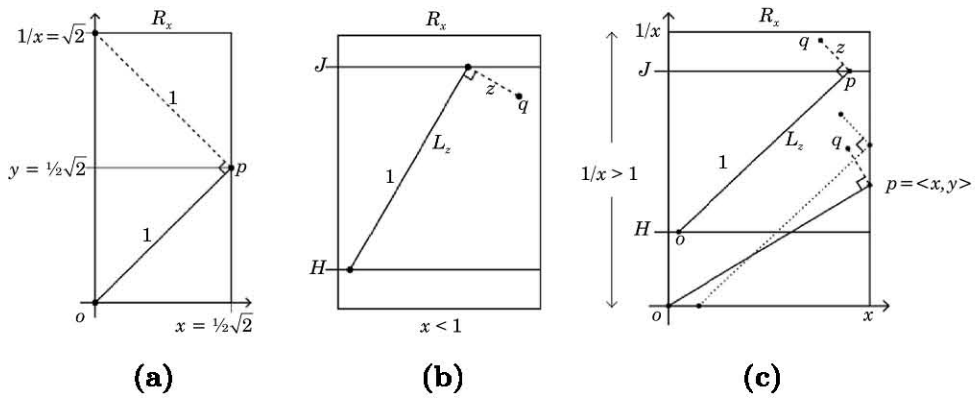


Fig. 9

q will fall on the same side of the vertical line through the vertex of the L_z as one of the vertical edges of R_x; call that the *left* edge, as depicted in figure 9c. Establish Cartesian coordinates with axes along those edges of R_x. Translate the L_z within R_x to obtain a congruent image (dotted) with vertex p on the right edge and the free end o of its unit segment on the bottom edge. Rotate that image within R_x so that p moves downward on the right edge, and o leftward until it reaches the bottom left corner; p will then have coordinates $\langle x, y\rangle$ with $0 < y \leq 1/x$ and $x^2 + y^2 = 1$. The task now is to determine, in this case, the constraints on z and x corresponding to the requirement that this copy of L_z lie in R_x.

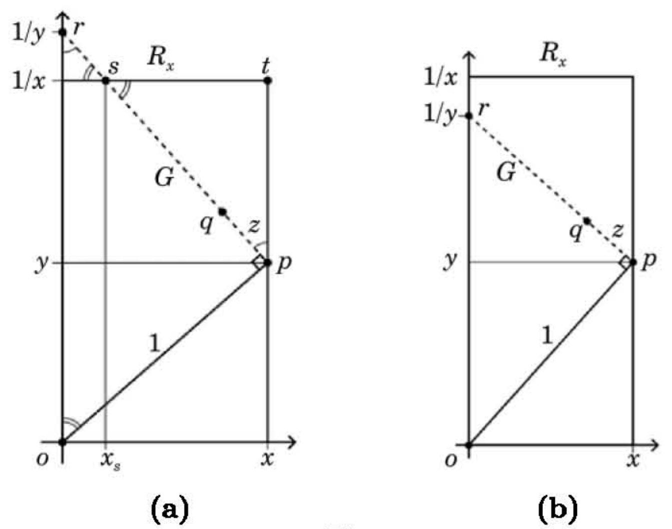


Fig. 10

Let G be the line $\overleftrightarrow{pq}$. Calculate its slope $-x/y$, vertical intercept $r = \langle 0, 1/y \rangle$, and distance

$$\overline{pr} = \sqrt{(0-x)^2 + \left(\frac{1}{y} - y\right)^2} = \sqrt{\frac{1}{y^2} - 1} = \frac{x}{y} = \frac{x}{\sqrt{1-x^2}} \, . \qquad (2)$$

If r should fall outside R_x as shown in figure 10a, then $1/x < 1/y$ and therefore $x > y$. Line G would intersect the top edge of R_x at a point $s = \langle x_s, 1/x \rangle$ with $0 < x_s < x$. Triangle $\Delta\, o\, p\, r$ would be similar to $\Delta\, s\, p\, t$, where t is the top right corner of R_x, and therefore

$$\frac{\overline{p\,s}}{\overline{t\,s}} = \frac{\overline{o\,r}}{\overline{o\,p}} \qquad\qquad \overline{p\,s} = \frac{\overline{t\,s} \cdot \overline{o\,r}}{\overline{o\,p}} = \frac{x - x_s}{y} \, .$$

Let $\mu = y/x$, so that $\mu < 1$ and $\mu^2 - \mu < 0$. By the slope-intercept equation for G and algebra,

$$x - x_s = \frac{y}{x}\left(\frac{1}{x} - y\right) = y\left(\frac{y^2 + x^2}{x^2} - \frac{y}{x}\right) = y\,(\mu^2 - \mu + 1) < y \, .$$

Thus, the L_z will lie inside R_x just when $z \le \overline{p\,s}$; and $\overline{p\,s} < 1$. In particular, no L_1 can lie inside R_x in this case.

If r should lie inside R_x as in figure 10b, then $1/x \ge 1/y$ and therefore $x \le y$ and $x/y \le 1$. An L_z will lie inside R_x in this position just when $z \le \overline{p\,r}$. In particular, an L_1 will lie inside R_x in this position just when $1 \le \overline{p\,r}$. Since $\overline{p\,r} = x/y$ by equation (2), this will happen only when $x/y = 1$, and thus only when $x = y = \frac{1}{2}\sqrt{2}$ and p and r are as shown in figure 9a: the proof of lemma 3.2 is complete. ♦

Lemma 3.3 *The edges in the path $\mathcal{P}$ described at the start of this section must intersect those in ∂Q at single internal points t and u of two opposite edges of Q. (For example, see figure 5.)*

Proof. The edges in $\mathcal{P}$ must intersect those in ∂Q at exactly two points: otherwise, $\mathcal{P}$ would not divide Q into just two subregions. If those points fell on adjacent edges, the other two edges would form an L_1 that would lie entirely in ∂A or ∂B. By lemma 3.2, its image under the isometry α or β would then be an $L_1 \subseteq R_x$ situated as in figure 9a and thus a part of $\partial A'$ or of $\partial B'$, and one of A' and B' would not be a polygonal subregion as required. ♦

Figure 11 is employed in showing that the dissections $Q = A \cup B$ and $R_x = A' \cup B'$ under consideration must be m-step staircases for some m. It suggests the orientation of the viewer but does not itself impose further constraints. The two opposite edges of Q mentioned in lemma 3.3 and the two edges of R_x of length x are depicted *horizontally*; the others are *vertical.*

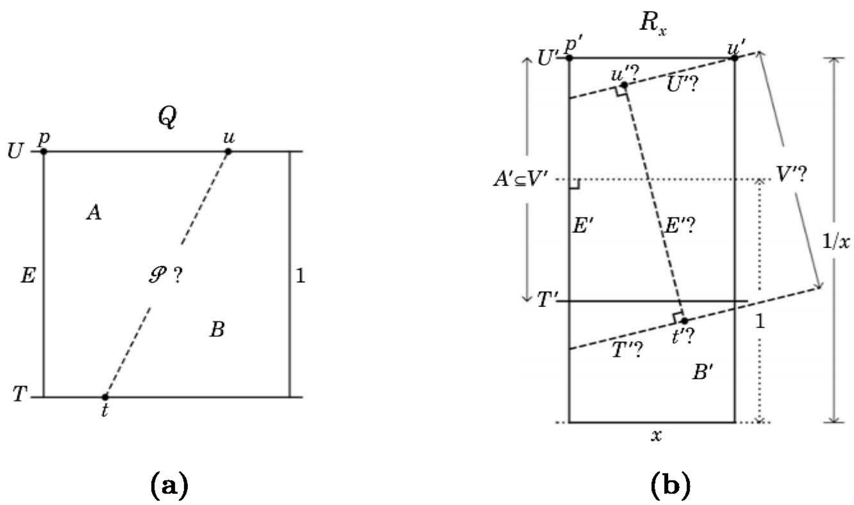


Fig. 11

LEMMA 3.4 *Each of $\partial A, \partial B$ contains a vertical edge of Q. Each of their images $\partial A', \partial B'$ under α, β contains a horizontal edge of R_x, lies in the horizontal strip one unit wide between that edge line and a parallel line, and contains a unit segment in a vertical edge of R_x. The image $\mathcal{P}'$ of the path $\mathcal{P}$ lies in both strips; therefore $x \geq \frac{1}{2}$.*

PROOF(extending over three paragraphs). The first statement follows from lemma 3.3. The isometry α maps the vertical edge E in A to a unit segment E' in R_x, which is too long to be horizontal. (Primes $'$ designate images under α. Images such as E' that are only tentatively placed in figure 11 are dotted or indicated by ? marks.) Since A lies between the horizontal edge lines T, U of Q, its image A' lies entirely in the strip V' between their images T', U', which are perpendicular to E' at its ends. Two opposite corners of R_x may not both fall outside V': otherwise, they would both fall outside A' and thus in B', and the isometry β^{-1} would map both into Q, contradicting lemma 3.1. If V' were oblique as depicted tentatively in figure 11b, it would exclude all four corners of R_x, so it must include just one horizontal edge of R_x. Moreover, A' must contain both ends of that edge: otherwise, B' would contain a pair of opposite corners.

There are just two points in $A \cap U$ that separate rays in U containing *no* other points of A from rays containing *some* other points of A: namely, the corner p and the end u of the path $\mathcal{P}$. Their images u' and p' in A' must therefore be the only two points in $A' \cap U'$ that separate rays in U' containing *no* other points of A' from rays containing *some* other points of A'. The corners of R_x in A' also satisfy this description; therefore, p' and u' must be those corners and A' must include that edge of R_x. Since E is the segment in ∂A that is perpendicular to U at p, its image E' is the segment in $\partial A'$ that is perpendicular to U' at p'.

The same reasoning can be repeated with β, B in place of α, A. A horizontal dotted line in figure 11b suggests the position of the strip in R_x that includes B'. Because the strips must overlap, $2\,({}^1\!/\!_x - 1) \le {}^1\!/\!_x$ —that is, $x \ge {}^1\!/\!_2$. ♦

The previous discussion distinguished *horizontal* and *vertical* features of figure 11 but not left, right, lower, or upper features. For displaying A and A' along horizontal and vertical edges of Q and R_x, *upper* and *left* were selected arbitrarily and tacitly. These terms are employed in the next section.

4. Construction. In figure 11b, the horizontal strips in R_x that include A' and B' must overlap, at least along a common borderline T'. If they have only T' in common, then segments $T' \cap R_x$ and $T \cap Q$ will constitute 1-step staircase paths that separate A', B' and A, B, respectively; moreover, $x = \frac{1}{2}$ and A, B will be rectangles with dimensions 1 by $\frac{1}{2}$. This proves

LEMMA 4.1 *In the special case of figure 12, the path $\mathcal{P}$ is a 1-step staircase.*

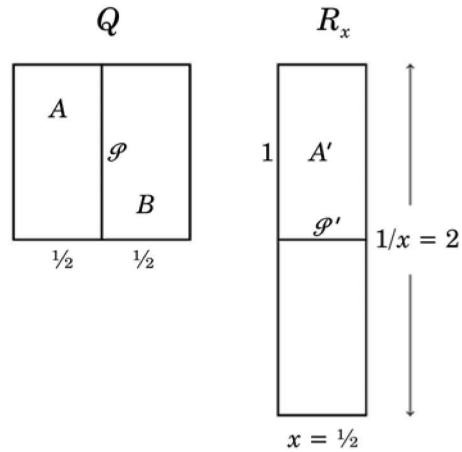


Fig. 12

Now turn to the general case, in which $x > \frac{1}{2}$. The horizontal strips mentioned in lemma 3.4 overlap as shown by dotted horizontal segments in figure 13.

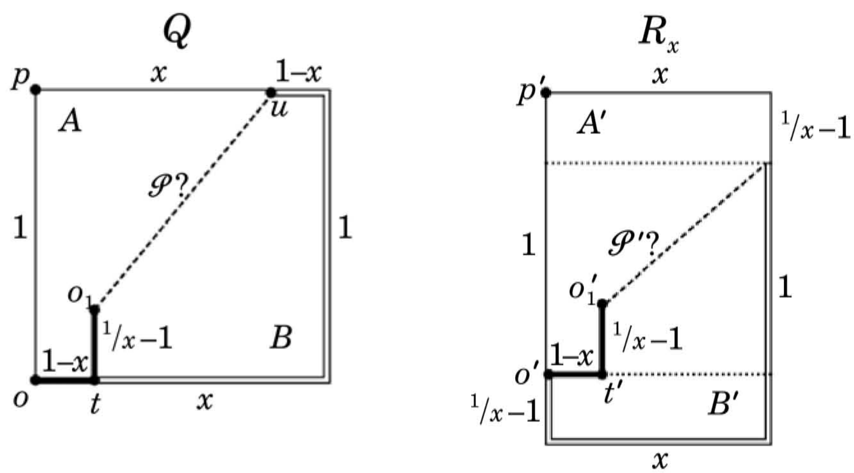


Fig. 13

Other features that have been established by the previous lemmas are indicated by single and double lines. The dashed lines suggest the paths $\mathcal{P}$ and $\mathcal{P}'$, yet to be determined.

By lemma 3.4, ∂A contains consecutive perpendicular edges of lengths x, 1, and $1-x$: segments $|up|$, $|po|$, and $|ot|$ (see figure 13). The same must be true of its image $\partial A'$ under α. The edge $|o't'|$ in $\partial A'$ of length $1-x$ contains points interior to $Q' = R_x$: it is an edge in the path $\mathcal{P}' = \partial_{\neg Q'} A' = \partial_{\neg Q'} B'$. Thus, $\partial B'$ contains consecutive edges whose lengths are 1, x, and $1/x - 1$; the same must be true of its image ∂B under β^{-1}. That edge $|t o_1|$ in ∂B of length $1/x - 1$ contains points interior to Q: it is an edge in the path $\mathcal{P} = \partial_{\neg Q} A = \partial_{\neg Q} B$. This proves

LEMMA 4.2 *Boundary edges of Q and R_x are apportioned among $\partial A, \partial B$ and $\partial A', \partial B'$ as shown in figure 13, which also describes with heavy single lines the initial edges $|ot|$ and $|o't'|$ of the paths $\mathcal{P}$ and $\mathcal{P}'$.*

The argument that the paths $\mathcal{P}$ and $\mathcal{P}'$ under consideration must adhere to Moese's staircase design is recursive. This paragraph describes their bottom and leftmost steps. Later it will be shown how successive steps are constructed until the resulting staircases $\mathcal{S}$ and $\mathcal{S}'$ reach the top and right edges of Q and $Q' = R_x$. The first step $\mathcal{S}_1$ of the staircase in Q consists of two segments already described: the horizontal *tread* $|ot|$ and vertical *riser* $|t o_1|$ shown with heavy lines in figure 13. According to lemma 4.2 they are both in ∂A; the riser is in $\mathcal{P}$. Since ∂A contains consecutive perpendicular edges of length 1, $1-x$, and $1/x - 1$, so does $\partial A'$: edges $|p' o'|$, $|o' t'|$, and $|t' o'_1|$, where $o'_1 = \alpha(o_1)$. The latter two edges are the tread and riser of the first step $\mathcal{S}_1{}' = \alpha[\mathcal{S}_1]$ of the staircase in $Q' = R_x$, also shown with heavy lines in figure 13. They contain points interior to R_x, and thus belong to the path $\mathcal{P}' = \partial_{\neg Q'} A' = \partial_{\neg Q'} B'$.

As figures 13 and 5 suggest, the paths $\mathcal{P}$ and $\mathcal{P}'$ will be determined by starting with these first steps and repeatedly appending congruent steps to the upper right ends of the growing staircases. Each tread will have width $1-x$ and each riser, height $1/x - 1$. Thus, k successive stairs will fit into the regions Q, R_x just when these equivalent conditions are satisfied:

$$k(1-x) \le x \qquad 1/x - 1 \le 1 \qquad k \le m, \text{ where } m = \left\lfloor \frac{x}{1-x} \right\rfloor \ge 1\,.$$ [9]

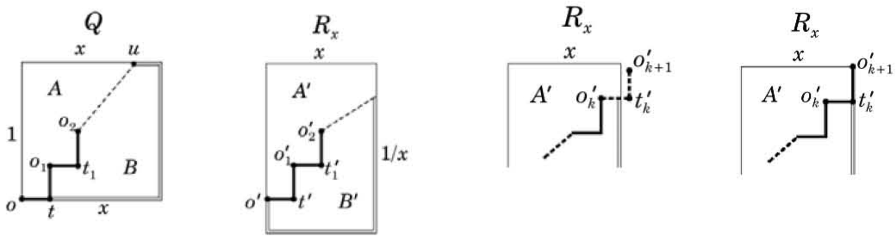
[9]The $\lfloor \ldots \rfloor$ notation stands for the *floor* function: the largest integer $\le$ the enclosed expression.

The composite isometry $\tau = \beta^{-1} \circ \alpha$ will play a major but tacit role in the recursive argument. Its values at the noncollinear points o, o_1, o'_1 agree with those of the translation by vector $\overrightarrow{o\,o'_1}$; therefore, it *is* that translation. The staircase in Q can be defined as

$$\mathcal{S} = \bigcup_{k=1}^{m-1} \mathcal{S}_k = \mathcal{S}_1 \cup \tau^1[\mathcal{S}_1] \cup \cdots \cup \tau^{m-1}[\mathcal{S}_1] .$$

However, α and β^{-1} figure individually in the recursion; τ will play no explicit role.

For $k = 1, \ldots, m$, repeat the argument in the rest of this paragraph. Define the points t_k and o_{k+1} and the $k+1^{\text{st}}$ step $\mathcal{S}_{k+1} = \beta^{-1}[\mathcal{S'}_k]$ consisting of the tread $|\,o_k\,t_k\,|$ and riser $|\,t_k\,o_{k+1}\,|$: see figure 14a-14b for the case $k = 1$. Since $\mathcal{S}'_k \subseteq \partial B'$, it follows that $\mathcal{S}_{k+1} \subseteq \partial B$. Since $k \leq m$, this tread and riser both contain points interior to Q. Thus, $\mathcal{S}_{k+1}$ lies in the path $\mathcal{P} = \partial_{\neg Q} A = \partial_{\neg Q} B$. Define points t'_k and o'_{k+1} and the $k{+}1^{\text{st}}$ step $\mathcal{S}'_{k+1} = \alpha[\mathcal{S}_{k+1}]$ consisting of the tread $|\,o'_k\,t'_k\,|$ and riser $|\,t'_k\,o'_{k+1}\,|$. Since $\mathcal{S}_{k+1} \subseteq \partial A$, it follows that $\mathcal{S}'_{k+1} \leq \partial A'$. Its tread contains points interior to R_x. If its riser does, too, then $\mathcal{S}'_{k+1}$ lies in the path $\mathcal{P}' = \partial_{\neg Q'} A' = \partial_{\neg Q'} B'$ and this recursive argument can continue.



(a) **(b)** **(c)** **(d)**

Fig. 14

If that riser $|\,t'_k\,o'_{k+1}\,|$ lay in the exterior of R_x —that is, if an additional recursive step would overshoot the right edge of R_x as in figure 14c—then $\mathcal{S}'_{k+1} = \alpha[\mathcal{S}_{k+1}]$ could not have lain in A', let alone in $\partial A'$, because α maps A to A'. Thus, the tread $|\,o'_k\,t'_k\,|$ of step $\mathcal{S}'_k$ must have ended at the right edge of R_x as in figure 14d. That is, $k(1-x) = x$, $k\,({}^1\!/\!_x - 1) = 1$, $k = m$, and the path $\mathcal{P}$ ends on the top edge of Q. By lemma 4.2, it ends at the point u. Moese's conjecture has thus been verified:

THEOREM 4.3 *For any dissections* $Q = A \cup B$ *and* $Q' = A' \cup B'$ *of square and nonsquare rectangles into polygonal subregions with disjoint interiors, such that* $A \cong A'$, $B \cong B'$, *and such that the sets* $\mathcal{P} = \partial_{\neg Q} A = \partial_{\neg Q} B$ *and* $\mathcal{P}' = \partial_{\neg Q'} A' = \partial_{\neg Q'} B'$ *are polygonal paths, the sets* $P \cup \partial Q$ *and* $\mathcal{P}' \cup \partial Q'$ *are congruent to the corresponding sets for one of Moese's* m*-step staircase dissections.*

5. Historical and Cultural Notes. This section includes biographical sketches of the mathematicians who originated the study reported here, describes the historical context of their work, and discusses a connection with Tarski's seminal research on the definability of sets of real numbers. It is mostly adapted from the biographical material in the book [11] about Tarski's early life and work. For further information and references, consult that.

***Parametr* and *Młody matematyk*.** As noted earlier, Polish mathematicians lobbied in the 1920s for greater support for secondary-school mathematics. In a report [15] to the Polish teachers' union, Tarski proposed an assembly of research mathematicians and educators to discuss problems of instruction. In 1930, the government administrator, curriculum developer, and entrepreneur Antoni M. Rusiecki founded the monthly journal *Parametr* for teachers and their best students. In 1931 Rusiecki started its supplement, *Młody matematyk* (*Young Mathematician*) specifically for students. The journal was impressive, but largely a one-man show. Its 1930 volume included 103 articles and notes. Rusiecki contributed 45 of them; 4 others were due to Tarski. The first two volumes included 140 problems, with 76 by Rusiecki and 14 by Tarski. In every issue, Rusiecki complained about overwork! This project ran out of energy and finances after two years. It published another volume in 1939, but World War II brought it to a halt. The high quality of this work and its role as a forerunner of post-War educational development in Poland is featured in reference [11] and the 2019 dissertation [4] of Ewa Dabkowska. Its history warrants further study.

Henryk Moese was born in 1886 in southern Poland. He began a teaching career around 1910, probably at a private secondary school in Kolbuszowa, a small town then in the Austrian Empire. In 1919 he moved to Śrem, a small city in western Poland, to become a teacher of mathematics, physics, and geography at the public school. By 1929 he had become its assistant director. In 1930 Moese was appointed director of the publicly funded classical liceum in Kępno, at that time near the German border. It enrolled about 275 students, about one-fourth


TOM 2 ZESZYT 8–10

PARAMETR

CZASOPISMO POŚWIĘCONE NAUCZANIU MATEMATYKI
Z DODATKIEM „MŁODY MATEMATYK"
WYCHODZI POD REDAKCJĄ A. M. RUSIECKIEGO
PRZY WSPÓŁUDZIALE DRA STEFANA STRASZEWICZA

PAŹDZIERNIK – GRUDZIEŃ 1932

ZAPISKI:

KAZIOR B. Ciekawy sposób dzielenia przez 9 298
Ł. Równanie ogólne stopnia drugiego z dwiema zmiennemi 298
MOESE H. Przyczynek do problemu A Tarskiego: „O stopniu równoważności wielokątów" . . . 305
TARSKI A. Uwagi o stopniu równoważności wielokątów 310

ROK 1 MARZEC 1931 NR. 3

MŁODY MATEMATYK

CZASOPISMO DLA MŁODZIEŻY SZKOLNEJ
WYCHODZI POD REDAKCJĄ A. M. RUSIECKIEGO
PRZY WSPÓŁUDZIALE S. STRASZEWICZA.

TREŚĆ: J. Gadomski. Odkrycie Plutona. — A. Tarski. O stopniu równoważności wielokątów. — Rozwiązania zadań Nr. 40 i 57. — Zadania Nr. 167 — 181.

Fig. 15. *Parametr* and *Young Mathematician*: where this study started.

young women. Moese was responsible for a faculty of about twelve and taught mathematics himself for six hours a week to the upper three classes. During the early 1930s he contributed to the *Parametr* and *Młody matematyk* journals that were just described, and was also involved in the Boy Scout association. In 1933 he moved on and by September 1935 was serving as director of the Mikołaj Kopernik Gimnazjum in Toruń, the major city on the Vistula river midway between Warsaw and the Baltic.[10]

Adolf Lindenbaum (figure 6) was born in Warsaw in 1904, to a Jewish family that became involved in the motion-picture business; he was independently wealthy. After graduating from gimnazjum in 1922, Lindenbaum entered the University of Warsaw to study mathematics. He was active in student organizations and left-wing politics and earned the doctorate in 1927 under the supervision of Wacław Sierpiński. Over the next decade, Lindenbaum made many contributions to the research seminar of Jan Łukasiewicz and Alfred Tarski, including the notion of degree of equivalence featured in the present paper. Lindenbaum published about twenty papers on general topology, set theory, and mathematical logic. Several especially significant ones were coauthored with Tarski or Tarski's student Andrzej Mostowski. Some now-fundamental concepts in logic are named for Lindenbaum. He married another Warsaw logician, Janina Hosiassonówna. During 1940–1941, under the Soviet occupation, he taught at the pedagogical institute in Białystok. Adolf and Janina were murdered near Vilnius after the 1941 German invasion.

[10]The authors have not yet traced Moese's life before or after this period. His photograph in figure 4 was taken in Toruń.

Zenon Waraszkiewicz (figure 7) was born in 1909 in Warsaw, then part of the Russian Empire. His parents were schoolteachers. The family took refuge in Odessa during World War I, then returned to Warsaw. Zenon completed secondary school there in 1926 and entered the University of Warsaw to study mathematics. He earned the doctorate in 1932, supervised by Stefan Mazurkiewicz, and continued research in point-set topology and analysis. Until World War II Waraszkiewicz taught in Warsaw secondary schools and served as assistant at the Warsaw Polytechnic University and dozent at the University of Warsaw. During the German occupation, he taught in the Polish clandestine schools. In 1945 Waraszkiewicz became a professor at the new University of Łódz but died there that same year.

Fig. 16. Alfred Tarski in 1968

Alfred Tarski was born in 1901 to a Jewish mercantile family in Warsaw. He attended secondary school there under the German occupation during World War I and entered Warsaw University in 1918, soon after it was re-established as a Polish institution. He studied with mathematicians Mazurkiewicz and Sierpiński and logicians Łukasiewicz, Tadeusz Kotarbiński, and Stanisław Leśniewski, and quickly achieved major status as a researcher. Tarski's early publications, especially with Stefan Banach applying set theory to the study of area and volume, were groundbreaking. Tarski entered academic life just after the newly independent Poland had greatly expanded its university faculties: hardly any positions remained open. He never obtained full-time university employment in Poland, just part-time work in Warsaw as a lecturer and research supervisor. Tarski was employed full-time as a secondary-school teacher. He contributed to the educational journals mentioned earlier, and during three summers presented courses on logic to in-service teachers. With two colleagues, he coauthored a secondary-school geometry text that considerably improved on Wojtowicz's text [21], which he had been using.[11]

Burdened with those jobs, Tarski nevertheless attained world ac-

[11]For a detailed description and translations of substantial portions of the textbook [2] by Zygmunt Chwiałkowski, Wacław Schayer, and Tarski, see [11, §9.9 and chapter 13].

claim as a researcher in set theory and logic. From his seminar stemmed many results published during 1930–1960 that are now fundamental. The next note in this section describes an example. At the onset of World War II in September 1939, Tarski was attending a conference of the Unity of Science movement at Harvard and was stranded there. His wife and children survived the War by hiding; all other family members were murdered during the Nazi occupation. After several unsettled years, Tarski obtained a professorship at the University of California, Berkeley. There he founded what would become the world's leading center of research in logic. Around 1970 he lectured to general audiences on the subject of this article. The present authors attended.

Sidelight on Definability. Tarski noticed that Moese's result and the theorem that verifies his conjecture can be formulated in real arithmetic:

$$(\forall x \in \mathbb{R})\Big(\big(\exists m \in \mathbb{N}\big)\, x = \frac{m}{m+1} \iff \big(\,(0 < x < 1)\ \&\ (\varphi_1(x) \vee \varphi_2(x) \vee \cdots)\,\big)\Big) \tag{3}$$

$\forall, \exists$	for all, for some	$=, <$	equals, less than	$\in$ ...	belongs to
$\iff$	if and only if	$+, -, \times, /$ ...	arithmetic operations	$\mathbb{R}$ ...	real numbers
$\implies$	implies			$\mathbb{N}$ ...	positive integers
$\&, \vee, \neg$ ...	and, or, not	$0, 1$	constants		

For each k, the statement φ_k will express the existence of a k-step staircase decomposition as described earlier, in terms of Cartesian coordinates. It will use only the symbols in the left-hand and center columns of the displayed glossary, parentheses, and variables such as x for real numbers, but *not* the ellipsis $\cdots$. With a real variable m, statement (3) is equivalent to

$$(\forall m > 0)\left(m \in \mathbb{N} \iff \left(\varphi_1\Big(\frac{m}{m+1}\Big) \vee \varphi_2\Big(\frac{m}{m+1}\Big) \vee \cdots\right)\right).$$

This could be interpreted as a definition of $\mathbb{N}$ in terms of real arithmetic, if the underlying logic supported the infinite disjunction $\varphi_1(x) \vee \varphi_2(x) \vee \cdots$. That, however, is a nonstandard logical feature. It is essential here because there can be no upper bound for m in terms of x: if $x \to 1$, then $m \to \infty$. In fact, Tarski's research in Warsaw had shown that $\mathbb{N}$ *cannot* be defined in terms of real arithmetic using standard logic that supports only finite disjunctions.[12]

[12] In work completed in the 1920s but not published in full until decades later, Tarski had formulated an axiom system for real arithmetic using only the standard logical features just mentioned. See his research papers [16, page 233] and [19, pages 48–50, 53]. In later years, Tarski and his students and colleagues investigated infinitary logic deeply. Tarski discussed the material in this sidelight in his 1970 lecture.

Author Contributions: Both authors equally contributed to the conceptualization, methodology, formal analysis, investigation, and writing–original draft preparation.

Conflicts of Interest: The authors declare no conflict of interest.

Bibliography

[1] U. Amaldi. Article 3, Sui concetti di retta e di piano, and Article 6, Sulla teoria della equivalenza, 1912. In Enriques [8], volume 1, 37–91, 145–198. Revisions of the 1900 original versions in [6]. The 1911 first volume of [7] contains German translations.

[2] Z. Chwiałkowski, W. Schayer, and A. Tarski. *Geometrja dla trzeciej klasy gimnazjalnej.* Państwowe Wydawnictwo Książek Szkolnych, Lwów, 1935. The title and publisher's name mean *Geometry for the Third Gimnazjum Class* and "National School Books Publisher". Substantial portions are translated in [11], chapter 13.

[3] R. Courant and H. Robbins. *What Is Mathematics: An Elementary Approach to Ideas and Methods.* Oxford University Press, London, New York, and Toronto, 1941. There are several later editions. Zbl 2501977, JFM 67.0001.05.

[4] E. Dabkowska. *Polish Mathematics Education Periodicals from 1930 to 1950.* PhD thesis, Columbia University, New York, 2019. The author is also known as Ewa Stelmach.

[5] S. L. Devadoss and J. O'Rourke. *Discrete and Computational Geometry.* Princeton University Press, Princeton, 2011. ISBN 978-0-691-14553-2; 978-1-400-83898-1. Zbl 1232.52001.

[6] F. Enriques, editor. *Questioni riguardanti la geometria elementare.* Zanichelli, Bologna, 1900. Enriques [7] is a German translation; [8] is a revised and augmented edition. Zbl 2664851. JFM 31.0085.05.

[7] F. Enriques, editor. *Fragen der Elementargeometrie*, volume 1: *Die Grundlagen der Geometrie.* Zanichelli, Bologna, 1911. Translated and edited by Hermann Thieme, incorporating revisions drafted for Enriques [8]. Translation of Enriques [6]. Contains a translation of Amaldi [1]. Zbl 2630020. JFM 42.0502.01.

[8] F. Enriques, editor. *Questioni riguardanti le matematiche elementari*, volume 1: *Critica dei principii.* Zanichelli, Bologna, 1912. Revised and augmented edition of Enriques [6]. Contains Amaldi [1]. Zbl 2624364. JFM 43.0060.01.

[9] F. Enriques and U. Amaldi. *Elementi di geometria ad uso delle scuole secondarie superiori.* Zanichelli, Bologna, sixth edition, 1903. Originally published in 1900, this appeared in many editions, tailored to various audiences. Zbl 2657183. JFM 34.0566.06.

[10] J. Erickson. The Jordan polygon theorem. jordan-polygon-theorem.pdf, 2009. 4 pages, downloaded March 2021.

[11] A. McFarland, J. McFarland, and J. T. Smith, editors. *Alfred Tarski: Early Work in Poland—Geometry and Teaching.* Birkhäuser/Springer, New York, 2014. ISBN 978-1-4939-1473-9; 978-1-4939-1474-6. doi: 10.1007/978-1-4939-1474-6. Zbl 1310.01002.

[12] H. Moese. Przyczynek do problemu A. Tarskiego: 'O stopniu równoważności wielokątów'. *Parametr*, 2:305–309, 1931–1932. Translated as "Contribution to A. Tarski's problem 'On the degree of equivalence of polygons' " in [11, §7.2, 144–151].

[13] J. C. Stone and V. S. Mallory. *New Plane Geometry.* Sanborn, Chicago, 1937.

[14] A. Tarski. O równoważności wielokątów. *Przegląd matematyczno-fizyczny*, 2:47–60, 1924. Translated as "On the equivalence of polygons" in [11, chapter 5, 77–91].

[15] A. Tarski. Zjazd matematyków. *Ogniwo: organ Związku Zawodowego Nauczycielstwa Polskich Szkół Średnich i biuletyn zarządu głównego Z. Z. N. P. S. Śr.*, 9:401–402, 1929. The journal title means *Link: Organ of the Trade Union of Polish Secondary-School Teachers and Bulletin of the General Administration* (of the same organization). Translated as "An assembly of mathematicians" in [11, chapter 10, 225–228].

[16] A. Tarski. Sur les ensembles définissables de nombres réels. I. *Fundamenta Mathematicae*, 17:210–239, 1931. ISSN 0016-2736. doi: 10.4064/fm-17-1-210-239. Translated as "On definable sets of real numbers" in [20, chapter VI, 110–142]. There is no part II. Zbl 2555102, JFM 57.0060.02.

[17] A. Tarski. O stopniu równoważności wielokątów. *Młody matematyk*, 1:37–44, 1931–1932. Reprinted in 1975 in *Delta* 5(4): 3–6. Translated as "On the degree of equivalence of polygons" in [11, §7.2, 135–144].

[18] A. Tarski. Uwagi o stopniu równoważności wielokątów. *Parametr*, 2:310–314, 1931–1932. Translated as "Remarks on the degree of equivalence of polygons" in [11, §7.4, 151–158].

[19] A. Tarski. *A Decision Method for Elementary Algebra and Geometry. Report R-109.* Technical report, RAND Corporation, Santa Monica, California, 1948. Prepared for publication by J. C. C. McKinsey. Zbl 0035.00602.

[20] A. Tarski. *Logic, Semantics, Metamathematics: Papers from 1923 to 1938.* Hackett, Indianapolis, second edition, 1983. ISBN 0-915144-75-1, 0-915144-76-X. Translated by J. H. Woodger. Edited with introduction and analytical index by J. Corcoran. Zbl 0075.00702.

[21] W. Wojtowicz. *Zarys geometrji elementarnej do użytku szkół średnich.* Wydawnictwo Zakładu Narodowego im. Ossolińskich, Lwów, sixth edition, 1926. The first edition was published in 1919 in Warsaw by Gebethner i Wolff. The title means *Outline of Elementary Geometry for Use in Secondary Schools.* The third edition has been revised and modernized by J. Norwa and published in the series Gra∂ient–Kufer Matematyków in 2000–2001 in Warsaw by the Fundacja Rozwoju Matematyki Polskiej.

Schody: *Jedyny* sposób na przejście

J. Shilleto i James T. Smith

Streszczenie. W płaskiej geometrii euklidesowej dowolne obszary wielokątne Q, Q' o równej powierzchni można podzielić na tę samą liczbę n parami przystających podregionów. Łatwo sprawdzić, czy Q jest kwadratem jednostkowym, Q' jest prostokątem $(m+1)/m$ na $m/(m+1)$ dla niektórych liczb całkowitych m, i $n = m+1$. W latach dwudziestych XX wieku znany logik Alfred Tarski, wówczas nauczyciel podstawowej geometrii w Warszawie, zauważył inny podział tego samego Q, Q' z $n = 3$. Opisał to w pionierskim czasopiśmie dla nauczycieli i uczniów szkół średnich, po czym zapytał, czy istnieją inne takie podziały dla $n = 2$? Henryk Moese, nauczyciel, odpowiedział tak, w niektórych przypadkach: m-"schodowe" cięcia są możliwe dla tych wymiarów. Moese przypuszczał, że *tylko* takie sekcje Q, Q' są tymi, o które padło pytanie. Tarski poinformował, że zostało to potwierdzone, ale dowód był zbyt skomplikowany, aby go opublikować. Kilkadziesiąt lat później Tarski przedstawił ten materiał szerokiej publiczności akademickiej, w tym także obecnym autorom. Potraktowaliśmy przypuszczenie Moese'a jako wyzwanie, opracowaliśmy sposób na jego weryfikację i przedstawiliśmy tutaj ten argument.

Używając jedynie elementarnej geometrii, z wieloma figurami, można przypuszczać, jakich metod Tarski mógł się spodziewać przy rozwiązywaniu innych problemów w tym czasopiśmie, ale nie przypomina żadnych argumentów w dostępnej literaturze. Opisujemy informacje o polskich wysiłkach podejmowanych w latach trzydziestych XX wieku na rzecz poprawy nauczania w szkołach średnich oraz o roli seminarium badawczego Tarskiego.

2010 Klasyfikacja tematyczna AMS (2010): 01A60 (01A70, 01A73).

Słowa kluczowe: Matematyka i matematycy XX wieku w Polsce, Uniwersytet Józefa Piłsudzkiego w Warszawie, Alfred Tarski, *Parametr*, elementarna geometria.

J. Shilleto
Richmond, CA, USA
E-mail: `jim.calculustutor@gmail.com`

James T. Smith
San Francisco State University
San Francisco, CA, USA
E-mail: `smith@sfsu.edu`

Communicated by: Stanisław Domoradzki

(Received: 10 sierpnia 2022; revised: 23 grudnia 2023)